REMARKS ON AN INVERSE BOUNDARY VALUE PROBLEM

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The problem of imaging the interior of the earth arises naturally in geophysics [Cl]. The problem of finding a method for electrical prospection based on just measurements at the boundary in order to determine the impedance (conductivity) of the earth's interior was proposed in the case that the impedance depend only on depth by Slichter [S] and it was first considered and formulated in generality by Calderón [(C]). Electrical impedance methods have also been used to measure certain cardiac parameters [H-W] and, in general, arises as a natural way of determining conductivity contrasts in the human body. We formulate now more precisely the mathematical problem.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$ and let γ be a strictly positive function in $L^\infty(\overline{\Omega})$. (We shall assume throughout $\gamma \in C^\infty(\overline{\Omega})$.) We consider the differential operator defined by

$$L_\gamma u = \text{div }(\gamma \nabla u) = \gamma \Delta u + \nabla \gamma \cdot \nabla u.$$

We solve the Dirichlet problem, given $\varphi \in H^{1/2}(\partial\Omega)$, find u solving

$$\begin{aligned} L_\gamma u &= 0 \quad \text{in} \quad \Omega \\ u \big|_{\partial\Omega} &= \varphi. \end{aligned} \tag{1}$$

We associate to u as in (1) its Dirichlet integral

$$Q_\gamma(\varphi) = \int_\Omega \gamma |\nabla u|^2. \tag{2}$$

γ is called here the *conductivity* of Ω ($\frac{1}{\gamma}$ measures the resistivity of Ω) and $Q_\gamma(\varphi)$ measures the power needed to maintain a potential φ on the boundary. The problem, proposed by Calderón, is

* Supported by NSF grant DMS-8600797

** Supported by NSF grant DMS-8601118 and an Alfred P. Sloan Research Fellowship

whether the measurement of $Q_\gamma(\varphi)$ for all $\varphi \in H^{1/2}(\partial\Omega)$ determines γ in Ω, i.e., is the map

$$\gamma \xrightarrow{Q} Q_\gamma \quad \text{injective?} \tag{3}$$

By polarizing the quadratic form (2), knowing $Q_\gamma(\varphi)\forall\varphi \in H^{1/2}(\partial\Omega)$, determines

$$Q_\gamma(\varphi,\psi) = \int_\Omega \gamma\nabla u \cdot \nabla v \tag{4}$$

with u, v solutions of $L_\gamma w = 0$ in Ω with $u\big|_{\partial\Omega} = \varphi$, $v\big|_{\partial\Omega} = \psi$.

Using Green's formula in (4) we have

$$Q_\gamma(\varphi,\psi) = \int_{\partial\Omega} \gamma u \frac{\partial v}{\partial\nu} \tag{5}$$

where ν denotes outer unit normal at $\partial\Omega$. Therefore Q_γ determines a unique self adjoint map

$$\begin{aligned} \Lambda_\gamma : H^1(\partial\Omega) &\longrightarrow L^2(\partial\Omega) \\ u\big|_{\partial\Omega} &\longrightarrow \left(\gamma\frac{\partial u}{\partial\nu}\right)\Big|_{\partial\Omega}. \end{aligned} \tag{6}$$

$\left(\gamma\frac{\partial u}{\partial\nu}\right)\Big|_{\partial\Omega}$ measures the electrical flux density entering or leaving the boundary. Λ_γ is the Neumann map called also here the *voltage* to *current* map. Calderón's question can then be rephrased: is the map

$$\gamma \xrightarrow{\Lambda} \Lambda_\gamma \quad \text{injective?} \tag{7}$$

¿From the practical point of view Λ_γ involves only measurements at the boundary. For every potential on the boundary we measure the induced current.

Calderón [C] proved that the linearized map dQ is injective at the constants.

Theorem 1. *$dQ\big|_{\gamma=1}$ is injective.*

Proof. Let $\delta \in C_0^\infty(\Omega)$, and

$$\gamma(t,x) = 1 + t\delta(x). \tag{8}$$

Let $Q_{\gamma(t,\cdot)}$ be the curve of quadratic forms associated to $\gamma(t,\cdot)$ as in (2), parametrized by t. We have, as in (4),

$$Q_{\gamma(t,\cdot)}(\varphi,\psi) = \int_\Omega \gamma\nabla u \cdot \nabla v \tag{9}$$

where

$$\begin{aligned} L_\gamma u &= 0 \quad \text{in} \quad \Omega \\ u\big|_{\partial\Omega} &= \varphi \end{aligned} \quad , \quad \begin{aligned} L_\gamma v &= 0 \quad \text{in} \quad \Omega \\ v\big|_{\partial\Omega} &= \psi. \end{aligned}$$

We differentiate (9) with respect to t (we denote $\frac{\partial}{\partial t}$ by $\cdot$).

$$\dot{Q}_\gamma(\varphi,\psi) = \int_\Omega \dot{\gamma}\nabla u\cdot\nabla v + \int_\Omega \gamma\big(\nabla\dot{u}\cdot\nabla v + \nabla\dot{v}\cdot\nabla u\big). \tag{10}$$

We integrate the second term in (10) by parts and we obtain

$$\dot{Q}_\gamma(\varphi,\psi) = \int_{\dot\Omega} \dot{\gamma}\nabla u\cdot\nabla v + \int_{\partial\Omega} \gamma\Big(\dot{\varphi}\frac{\partial\psi}{\partial\nu} + \dot{\psi}\frac{\partial\varphi}{\partial\nu}\Big). \tag{11}$$

If we choose φ and ψ independent of t, we have

$$\dot{Q}_\gamma(\varphi,\psi) = \int_\Omega \delta\nabla u\cdot\nabla v. \tag{12}$$

Making use of the identity

$$L_\gamma(uv) = uL_\gamma v + vL_\gamma u + 2\gamma\nabla u\cdot\nabla v$$

and integrating by parts in (12), we get

$$\dot{Q}_\gamma(\varphi,\psi) = \int_\Omega L_\gamma\Big(\frac{\delta}{\gamma}\Big)uv. \tag{13}$$

In the case $t=0$, (13) becomes

$$\dot{Q}_1(\varphi,\psi) = \int_\Omega \Delta\delta uv \tag{14}$$

where

$$\begin{array}{ll} \Delta u = 0 \quad \text{in} \quad \Omega & \Delta v = 0 \quad \text{in} \quad \Omega \\ u\,\big|_{\partial\Omega} = \varphi & , \quad v\,\big|_{\partial\Omega} = \psi. \end{array} \tag{15}$$

Following Calderón [C] we choose complex plane wave solutions to (15)

$$\begin{array}{ll} u = e^{x\cdot\xi_1} & (\varphi = e^{x\cdot\xi_1}\,\big|_{\partial\Omega}) \\ v = e^{x\cdot\xi_2} & (\psi = e^{x\cdot\xi_2}\,\big|_{\partial\Omega}) \end{array} \tag{16}$$

with $\xi_j\in\mathbb{C}^n$, $j=1,2$, satisfying

$$\xi_j\cdot\xi_j = 0 \quad \text{for} \quad j=1,2, \quad \text{or if}$$

$$\xi_j = \eta_j + ik_j \tag{17}$$

$$|\eta_j| = |k_j|, \quad \eta_j\cdot k_j = 0, \quad j=1,2.$$

Calderón chose

$$\xi_1 = \frac{\eta}{2} - i\frac{k}{2}$$
$$\xi_2 = -\frac{\eta}{2} - i\frac{k}{2}$$

with $\eta, k \in \mathbf{R}^n$ satisfying (17). We get

$$\dot{Q}_1(e^{x\cdot\xi_1}\big|_{\partial\Omega}, e^{x\cdot\xi_2}\big|_{\partial\Omega}) = \int_\Omega \Delta\delta e^{-ix\cdot k} = \widehat{\Delta\delta}(k) \tag{18}$$

where $\widehat{\ }$ denotes the Fourier transform and we have extended δ to be zero outside Ω. This shows that if $\dot{Q}_1 = 0$ then $\widehat{\Delta\delta}(k) = 0\ \forall k$ and therefore $\Delta\delta = 0$ in Ω by the Fourier inversion formula and then $\delta = 0$ in Ω since $\delta\big|_{\partial\Omega} = 0$. □

Calderón's proof gives actually a left inverse. From (18) we have

$$\delta = (\Delta_D)^{-1}[\dot{Q}_1(e^{x\cdot\xi_1}\big|_{\partial\Omega}, e^{x\cdot\xi_2}\big|_{\partial\Omega})]^{\vee}$$

where $\vee$ denotes the inverse Fourier transform and $(\Delta_D)^{-1}$ is the solution operator to the Dirichlet problem. However the linearized map dQ is not onto and therefore the implicit function theorem cannot be applied to construct a local left inverse for Q. This difficulty was overcome in [S-U, I] to obtain a local uniqueness result in dimension 2 (Theorem 3). Furthermore in [S-U, II] it was obtained a global uniqueness result (Theorem 4) for $n \geq 3$. We shall describe briefly the main ideas in the proof below.

We use the following result at the boundary proved by Kohn and Vogelius ([K-V, I]), namely that knowledge of Q_γ (or Λ_γ) determines the Taylor series of γ at the boundary of Ω.

Theorem 2. *Let γ_i $(i = 0, 1)$ be $C^\infty(\overline{\Omega})$ with a positive lower bound. Let $x_0 \in \partial\Omega$ and let U be a neighborhood of x_0 relative to $\overline{\Omega}$. Suppose that*

$$Q_{\gamma_0}(\varphi) = Q_{\gamma_1}(\varphi)\ \forall \varphi \in H^{1/2}(\partial\Omega) \text{ with supp } \varphi \subseteq B \cap \partial\Omega,$$

then $\partial^\alpha\gamma_0(x_0) = \partial^\alpha\gamma_1(x_0)\ \forall\alpha$.

As a corollary of the theorem we see that a real analytic γ is a priori determined by Λ_γ. Kohn and Vogelius ([K-V ,II]) have extended this result to cover piecewise analytic γ.

Sketch of proof of Theorem 2.

A different sketch of the proof than that of Kohn and Vogelius who used elliptic regularity follows. It is well known that Λ_γ, the voltage to current map, is a classical pseudodifferential operator of order 1 on $\partial\Omega$. Its principal symbol $\sigma_{\Lambda_\gamma}(x,\xi) = \gamma(x)|\xi|$, and therefore knowing Λ_γ we can determine γ at the boundary and all of its tangential derivatives. Now, the ***full symbol*** of Λ_γ can be written asymptotically as an infinite sum of functions λ_k homogeneous of degree $1-k$. $\lambda_k(x,\xi)$ involves the normal derivative of γ of order k at x with a non-zero coefficient plus terms involving normal derivatives of order strictly less than k at x and tangential derivatives of order at most k. Then an inductive argument proves that we can determine all the derivatives of γ at the boundary from the full symbol of Λ_γ. □

Theorem 3. *[S-U, II]. Let $n \geq 3$, $\gamma_0, \gamma_1 \in C^\infty(\overline{\Omega})$ with a positive lower bound so that*

$$Q_{\gamma_0} = Q_{\gamma_1}.$$

Then

$$\gamma_0 = \gamma_1 \quad \textit{in} \quad \overline{\Omega}.$$

Sketch of proof. Let γ be any smooth strictly positive function in $\overline{\Omega}$. One of the main ideas is to construct solutions of $L_\gamma u = 0$ in Ω of the form

$$u = e^{x\cdot\xi}\gamma^{-1/2}(1+\psi(x,\xi)) \tag{19}$$

where $\xi \in \mathbb{C}^n$ with $\xi\cdot\xi = 0$ as in Calderón's computation (Theorem 1). Using ideas from geometrical optics we would like that the solutions (19) behave like the complex plane waves $e^{x\cdot\xi}$ for $|\xi|$ large.

We want, then,

$$\begin{aligned} \psi(x,\xi) &\longrightarrow \quad 0 \\ \text{as} \quad |\xi| &\to \infty \end{aligned} \tag{20}$$

uniformly in $\overline{\Omega}$.

The "transport equation" for ψ is the singular perturbation problem

$$\Delta\psi + \xi\cdot\nabla\psi - q\psi = q \text{ in } \Omega \tag{21}$$

where

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}.$$

However, if we give boundary conditions for (21) at $\partial\Omega$, ψ will not satisfy, in general, the decay condition (20). Actually, we would expect that the dominant term in (21) for large $|\xi|$ is $\xi\cdot\nabla\psi$. In dimension 3 if $\xi = \eta + ik$ with $\xi\cdot\xi = 0$, $\xi\cdot\nabla\psi$ is the Cauchy Riemann equation in the planes perpendicular to η and certainly we cannot then impose general boundary conditions on ψ. Unable to characterize the boundary values of ψ satisfying (21) and (20) (this remains an interesting open question for reconstruction) we extended γ suitable and we look for solutions of (21) in the whole space and with growth conditions at infinity in the x-variable. We proved in [S-U, II].

Lemma 1. *Let $\gamma \in C^\infty(\mathbf{R}^n)$, $n \geq 3$ with γ strictly positive and $\gamma = 1$ outside a large ball containing Ω. Then there is a unique solution $\psi \in L^2_\delta$, $-1 < \delta < 0$ of*

$$\Delta\psi + \xi \cdot \nabla\psi - q\psi = q \text{ in } \mathbf{R}^n$$

satisfying

$$||\psi||_{H^s_\delta} \leq \frac{C}{|\xi|}, \quad s > \frac{n}{2}$$

with C depending on s, Ω, δ, q and H^s_δ is the weighted Sobolev space built over the weighted L^2_δ space with

$$||\psi||^2_{L^2_\delta} = \int_{\mathbf{R}^n} (1 + |x|^2)^\delta |\psi(x)|^2 dx.$$

Now we proceed with our sketch of proof of Theorem 3. Let

$$\gamma(t, x) = (1 - t)\gamma_0 + t\gamma_1 \text{ in } \Omega,\ 0 \leq t \leq 1 \tag{22}$$

$$\tilde{\gamma}(t, x) = \tilde{\gamma}_0 \quad \text{in} \quad \complement\Omega$$

where $\tilde{\gamma}_0$ is a smooth extension of both γ_0 and γ_1 (this is possible by Theorem 2) with $\tilde{\gamma}_0 = 1$ outside a ball that contains Ω. We consider solutions $L_\gamma u = 0$, $L_\gamma v = 0$ in $\mathbf{R}^n$ with γ as in (22) of the form (using Lemma 1)

$$\begin{aligned} u(x, \xi_1, t) &= e^{x\cdot\xi_1}\gamma^{-1/2}(1 + \psi(x, \xi_1, t)) \\ v(x, \xi_2, t) &= e^{x\cdot\xi_2}\gamma^{-1/2}(1 + \psi(x, \xi_2;, t)) \end{aligned} \tag{23}$$

with $\xi_k \cdot \xi_k = 0$, $k = 1, 2$, and Re $(\xi_1 + \xi_2) = 0$. We have (this is completely analogous to the computation made before with γ as in (8)).

$$\dot{Q}_\gamma(u\,|_{\partial\Omega}\,,\ v\,|_{\partial\Omega}) = \int_\Omega \dot{\gamma}\nabla u \cdot \nabla v + \int_{\partial\Omega} \gamma\Big(\dot{u}\frac{\partial v}{\partial\nu} + \dot{v}\frac{\partial u}{\partial\nu}\Big) \tag{24}$$

The difference is now that u and v depend on t at the boundary. However we have (see [S-U, II]).

Lemma 2. $u(x,\xi_k,0) = u(x,\xi_k,1) \forall x \in \complement\Omega$, $k = 1,2$. *The proof uses the fact that* $\gamma(\cdot,t)$ *is independent of* t *in* $\complement\Omega$ *and the fact that the Neumann map for* $\gamma(\cdot,0)$ *is equal to the Neumann map for* $\gamma(\cdot,1)$.

Using Lemma 2 and integration by parts we can write the boundary integral in (24) as an integral over a large ball. The growth condition on ψ at infinity (see [S-U, II] for more details) gives:

Lemma 3. *Let* u, v *be as in (23). Then*

$$\int_{\partial\Omega} \gamma\Big(\dot{u}\frac{\partial v}{\partial \nu} + v\frac{\partial \dot{u}}{\partial \nu}\Big) = 0.$$

Integrating (24) in t and using the fact that the boundary values of $u(x,\xi_k,0)$ and $u(x,\xi_k,1)$, $k = 1,2$ are the same we obtain and "average linearization"

$$\int_0^1 \int_\Omega \dot{\gamma} \nabla u \cdot \nabla v = 0. \tag{25}$$

Proceeding as in step (12) to (13) we get

$$\int_0^1 \int_\Omega L_\gamma\Big(\frac{\dot{\gamma}}{\gamma}\Big) uv = 0. \tag{26}$$

Now we make special choices of ξ_1, ξ_2 in (23), namely

$$\begin{aligned} \xi_1 &= \varsigma + i\Big(\frac{k}{2} + r\eta\Big) \\ \xi_2 &= -\varsigma + i\Big(\frac{k}{2} - r\eta\Big) \end{aligned} \tag{27}$$

where $k \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $\eta, \varsigma \in \mathbb{R}^n$ satisfying

$$\langle k, \eta \rangle = \langle k, \varsigma \rangle = \langle \eta, \varsigma \rangle = 0,$$

$|\eta| = 1$, $|\varsigma|^2 = \frac{|k|^2}{4} + r^2$, so that $\xi_k \cdot \xi_k = 0$, $k = 1,2$.

The idea is that

$$e^{x\cdot(\xi_1+\xi_2)} = e^{ix\cdot k} \tag{28}$$

with ξ_1, ξ_2 as in (27). The right hand side of (28) is the exponential in the Fourier transform. However, for fixed k, $\psi(x,\xi_1,t), \psi(x,\xi_2,t)$ approach zero, uniformly in $\overline{\Omega}$, as r approaches infinity. The choice (27) is only possible in dimension three or larger.

Now (26) becomes

$$0=\int_0^1 dt\int_{\mathbf{R}^n}\frac{1}{\gamma}L_\gamma\Big(\frac{\dot\gamma}{\gamma}\Big)e^{ix\cdot k}(1+\psi(x,\xi_1,t))(1+\psi(x,\xi_2,t)).$$

Letting r approach infinity and applying Lemma 1 we obtain

$$0=\int_{\mathbf{R}^n}\Big[\int_0^1 dt\frac{1}{\gamma}L_\gamma\Big(\frac{\dot\gamma}{\gamma}\Big)\Big]e^{ix\cdot k}\quad\forall k\in\mathbf{R}^n.$$

Therefore

$$0=\int_0^1 dt[\Delta\log\gamma+\tfrac{1}{2}|\nabla\log\gamma|^2]^\bullet$$

and using the fundamental theorem of calculus

$$0=\Delta(\log\gamma_1-\log\gamma_0)+\tfrac{1}{2}[|\nabla(\log\gamma_1)|^2-|\nabla(\log\gamma_0)|^2]$$

or

$$0=\Delta(\log\gamma_1-\log\gamma_0)+\tfrac{1}{2}\nabla(\log\gamma_1+\log\gamma_0)\cdot\nabla(\log\gamma_1-\log\gamma_0),$$

a linear equation for $\log\gamma_1-\log\gamma_0$ which vanishes on $\partial\Omega$ by the Kohn-Vogelius result. The maximum principle applies to give

$$\log\gamma_1-\log\gamma_0=0\quad\text{in}\quad\Omega$$

or

$$\gamma_1=\gamma_0\quad\text{in}\quad\Omega$$

proving the theorem. □

The global uniqueness problem in the two dimensional case remains open at present. The difficulty arises since in this case the inverse problem is formally determined. For $n\geq 2$, the kernel of the Neumann map is a function in $\partial\Omega\times\partial\Omega$ depending on $2(n-1)$ variables. The function γ depends on n variables and $2n-2=n$ for $n=2$ and $2n-2>n$ for $n\geq 3$. This freedom for $n\geq 3$ was explained in the choices of ξ_1,ξ_2 as in (27). For $n=2$, we can construct solutions of the form (21). Lemma 1 is valid although the proof is different because the term $\xi\cdot\nabla\psi$ is actually a Cauchy-Riemann equation (see [S-U, I]). The other ingredients, Lemma 2 and the average linearization are also true. However in this case we also need a low frequency estimate which was proven essentially by Calderón ([C]) in the case γ close to a constant. We have (see [S-U, I]):

Theorem 4. *Let $\gamma_0, \gamma_1 \in C^\infty(\overline{\Omega})$ with a positive lower bound and*

$$Q_{\gamma_0} = Q_{\gamma_1}.$$

Then $\exists\ \varepsilon(\Omega)$ such that if

$$\|\gamma_i - 1\|_{C^8(\overline{\Omega})} < \varepsilon(\Omega),\ i = 0, 1,\ \text{then } \gamma_0 = \gamma_1\ .$$

We shall study further the transport equation (21) and derive new results (Theorem 5). From this point on we assume that $n = 2$.

Theorem 5. *Let $n = 2$, $\gamma_0, \gamma_1 \in C^\infty(\overline{\Omega})$ with a positive lower bound and*

$$Q_{\gamma_0} = Q_{\gamma_1}.$$

Then

$$\int_\Omega (q_0 - q_1)(w) w^m dw \wedge d\overline{w} = \int_\Omega (q_0 - q_1)(w) \overline{w}^m dw \wedge d\overline{w} = 0$$

for all m integers $m \geq 0$, where $q_i = \frac{\Delta \gamma_i^{1/2}}{\gamma_i^{1/2}}$ and $w = x_1 + i x_2 \in \mathbb{R}^2$.

In other words $(q_0 - q_1)$ is orthogonal in $L^2(\Omega)$ to the set of analytic and anti-analytic functions in $\mathbb{C}$. (Therefore, since $q_0 - q_1$ is real, orthogonal to the set of harmonic functions in Ω). We easily obtain

Corollary 1. *Suppose γ_0, γ_1 satisfy the conditions of Theorem 5, then*

$$\int_\Omega |\nabla \log \gamma_0|^2 = \int_\Omega |\nabla \log \gamma_1|^2.$$

Proof. Take $m = 0$ in Theorem 4. Then use

$$q_0 - q_1 = \Delta \log(\gamma_0 - \gamma_1) + |\nabla \log \gamma_0|^2 - |\nabla \log \gamma_1|^2 \quad \text{and integrate.} \qquad \square$$

Corollary 2. *Let γ_0, γ_1 be as in Theorem 5 with $\gamma_0 = C$, then $\gamma_1 = C$.*

Proof. Using Corollary 1, we have

$$\int_\Omega |\nabla \log \gamma_1|^2 = 0.$$

Therefore $\gamma_1 =$ constant, and since γ_0 coincides with γ_0 in the boundary, $\gamma_1 = \gamma_0 = C$. $\square$

This means that we can distinguish constants from their Neumann maps.

Before going into the proof of Theorem 5 we point out that the transport equation (21) for $n = 2$ can be factorized and we want to solve with $\psi \in L^2_\delta$, $-1 < \delta < 0$.

$$\overline{\partial}(\partial + (k_2 + ik_1))\psi - q\psi = q \text{ in } \mathbf{R}^n \tag{29}$$

where $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$ and $\xi = \eta + ik$, $\eta, k \in \mathbf{R}^2$.

For $|k|$ large, the dominant term in (29) is the Cauchy-Riemann operator $\overline{\partial}$ and of course, we cannot give general boundary condition in $\partial\Omega$. However, as was proved by Nirenberg and Walker [N-W], given $f \in L^2_{\delta+1}$, $\exists!\ u \in L^2_\delta$ solving $Lu = f$ in $\mathbf{R}^2$, where L represents ∂ or $\overline{\partial}$.

This is one of the reasons why looking for solutions of (29) in the whole space works.

We proved in [S-U, I]:

Lemma 4. *Given $-1 < \delta < 0$, there exists a constant $C(\delta)$ such that, if $q \in L^2_{\delta+1}$ and $|k| \geq C||q(1+|x|^2)^{1/2}||_{L^\infty}$ then there exists a unique solution to*

$$\Delta\psi + (k_1 + ik_2)\overline{\partial}\psi - q\psi = q$$

such that $\psi, \nabla\psi \in L^2_\delta$. Moreover, ψ may be written in the form

$$\psi(x,k) = a(x,k) + e^{-ix\cdot k}c(x,k)$$

where

$$a(x,k) = \frac{a_1(x)}{k_2 + ik_1} + \sum_{j=2}^{\infty} \frac{a_j(x,k)}{(k_2 + ik_1)^j}$$

and

$$c(x,k) = \sum_{j=1}^{\infty} \frac{c_j(x,k)}{(k_2 + ik_1)^j}$$

with

$$||c_j||_{H^1_\delta}, ||a_j||_{H^1_\delta} \leq c||q(1+|x|^2)^{1/2}||^{j-1}_{L^\infty}||q||_{L^2_{\delta+1}} \tag{30}$$

Now we are in a position to prove the theorem.

Proof of Theorem 5.

We can rearrange the series for ψ as in Lemma 4 in the following way:

$$\psi = \frac{a_1(x)}{k_2 + ik_1} + \frac{-\partial a_1}{(k_2 + ik_1)^2} + \sum_{j=2}^{\infty} \frac{a_j}{(k_2 + ik_1)^j} \tag{31}$$

$$+e^{-ix\cdot k}\Big(\frac{h}{(k_2+ik_1)^2}+\sum_{j=2}^{\infty}\frac{c_j}{(k_2+ik_1)^j}$$

where

$$c_1=-\frac{e^{ix\cdot k}\partial a_1}{k_2+ik_1}+\frac{h}{k_2+ik_1} \tag{32}$$

with

$$\partial h=e^{ix\cdot k}\partial^2 a_1$$

since the right hand side of (32) satisfies the same equation as c_1, namely

$$\partial c_1=e^{-ix\cdot k}\partial a_1.$$

Now a_1 is determined by solving

$$\partial a_1=q \tag{32b}$$

Now from ((31) and the property (30) of the c_j, a_j's, we deduce that

$$\psi(x,k)=\frac{a_1(x)}{k_2+ik_1}+O\Big(\frac{1}{|k|^2}\Big)\text{ for }|k|\text{ large} \tag{33}$$

where the lower order term in (33) is uniformly bounded for compact subsets of $\mathbb{R}^2$.

Let us denote ψ^0 and ψ^1 respectively as the ψ's associated with γ_0 and γ_1. We also denote by a_1^0, a_1^1 the first term in (33). Using now Lemma 2 (which is also valid for $n=2$) we get:

$$\psi^0=\psi^1\quad\text{in}\quad \complement\Omega$$

and therefore by (33)

$$a_1^0=a_1^1\quad\text{in}\quad \complement\Omega. \tag{34}$$

Now

$$\overline{\partial}(a_1^0-a_1^1)=q_0-q_1$$

and q_0-q_1 has compact support. Therefore

$$(a_1^0-a_1^1)(z)=\int\frac{(q_0-q_1)(w)}{z-w}dw\wedge d\overline{w} \tag{35}$$

with $z=x_1+ix_2$.

By (34)

$$(a_1^0 - a_1^1)(z) = \frac{1}{z} \sum_{n=0}^{\infty} z^n \int (q_0 - q_1)(w) w^n dw \wedge d\overline{w}$$

for $|z|$ large enough. Therefore we conclude

$$\int (q_0 - q_1)(w) w^n dw \wedge d\overline{w} = 0 \ \forall n.$$

Changing ξ to $\overline{\xi}$ in the transport equation (21) changes (29) to

$$\partial(\overline{\partial} + (k_2 - ik_1))\psi - q\psi = q$$

and the equation for the analog of a_1 in Lemma 4 is

$$\partial a_1 = q.$$

Repeating the argument above, one gets

$$(a_1^0 - a_1^1)(z) = \int \frac{1}{\overline{z - w}} (q_0 - q_1)(w) dw \wedge d\overline{w}$$

and therefore

$$\int \overline{w}^n (q_0 - q_1)(w) dw \wedge d\overline{w} = 0$$

for all n, thus proving the theorem. □

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